Oligopoly with Hyperbolic Demand and Capital Accumulation[1]

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Abstract

A dynamic approach is proposed for the analysis of the Cournot oligopoly game with hyperbolic demand, showing that the adoption of capital accumulation dynamics either *à la* Solow-Swan or *à la* Ramsey eliminate the indeterminacy problem characterising the static model when marginal costs are nil. It is proved that the steady state equilibria produced by both models are stable in the saddle point sense. Finally, it is also shown that the solutions of the corresponding feedback problems share analogous properties, although they cannot be fully characterised from an analytical standpoint.

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1 Introduction

Most of the existing literature on oligopoly theory (either static or dynamic) assumes linear demand functions, as this, in addition to simplifying calculations, also ensures both concavity and uniqueness of the equilibrium, which, in general, wouldn't be warranted in presence of convex demand systems (see Friedman, 1977; and Dixit, 1986, *inter alia*).

In particular, a classic textbook example of a market with convex demand is that where the demand function has a hyperbolic shape, this being a special case of a more general class of models based on isoelastic demand curves. As is well known, in such a case the maximum problem of a firm choosing the output level is indeterminate if marginal cost is nil, since the revenue generated by a hyperbolic demand (or, in general, a demand function whose price elasticity is equal to one in absolute value) is constant.[1]

The aim of this paper is to illustrate a way out of the aforementioned problem, offered by dynamic game theory. In particular, I want to show that indeterminacy may well disappear altogether as soon as the accumulation of capacity for production is taken into account. To this purpose, I propose two differential oligopoly games where firms accumulate physical capital (i.e., productive capacity) in order to supply the final good to consumers. The first one is based on the Solow (1956) - Swan (1956) capital accumulation dynamics with costly investments, while the second is based on the Ramsey dynamics, where capacity accumulation is accompanied by the implicit cost of intertemporal relocation of unsold output associated to the accumulation of productive capacity (as in Cellini and Lambertini, 1998, 2007).[2]

[1]Of course this does not apply in general to models with isoelastic market demand functions. See, e.g., Anderson and Engers (1992, 1994) and Cellini and Lambertini (2007).

[2]See also Calzolari and Lambertini (2006, 2007) for applications of these dynamics to

The Solow-Swan modelization entails that quantity is no longer one of the controls (as it is in the static case). This, in turn, has the relevant implication that the first order condition becomes independent of the marginal cost involved by the production of the final good and therefore the equilibrium is determinate irrespective of the level of the marginal cost itself. Conversely, in the Ramsey approach firms still choose sales, as in the static model, but output and sales do not coincide given the Ramsey capital accumulation dynamics. Hence, (i) the resulting first order condition depends on the shadow price attributed to an additional unit of capacity, and consequently (ii) the control (i.e., sales) dynamics depend on the dynamics of the co-state variable (amongst other things). These facts imply that the Ramsey game produces multiple steady states, one closely replicating the Cournot equilibrium of the corresponding static setup, the other being the Ramsey golden rule, which is not sensitive to either the shape of market demand or the level of marginal costs. This ultimately yields that, if marginal cost were nil, the industry would converge to the golden rule equilibrium in the long run. The main body of the analysis is carried out relying upon the open-loop solutions of both games. As a complement, the feedback problems are also briefly outlined, to show that similar properties also hold for the subgame (or Markov) perfect solutions, although these cannot be analytically characterised given that the models do not exhibit a linear quadratic form.

The remainder of the paper is structured as follows. Section 2 summarises the static game. The Solow-Swan game is investigated in section 3. The Ramsey game is accounted for in section 4. The feedback approach is briefly outlined in section 5. Section 6 contains some concluding remarks.

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intraindustry trade and the related policy issues dealing with the optimal design of import quotas, tariffs, and voluntary export restraints.

2 A summary of the static game

As a preliminary step, it is useful to reconstruct the features of the static Cournot game with hyperbolic demand. N firms supply individual quantities q_i, $i = 1, 2, 3, ...N$. The good is homogeneous, and market demand is $p = a/Q$, $Q = \sum_{i=1}^{N} q_i$. Production entails a constant marginal cost $c \in [0, a)$. Market competition takes place *à la* Cournot-Nash; therefore, firm i chooses q_i so as to maximise profits $\pi_i = (p - c)\, q_i$. This entails that the following first order condition must be satisfied:

$$\frac{\partial \pi_i}{\partial q_i} = \frac{a \sum_{j \neq} q_j}{\left(q_i + \sum_{j \neq} q_j\right)^2} - c = 0 \tag{1}$$

and the associated second order condition:

$$\frac{\partial^2 \pi_i}{\partial q_i^2} = -\frac{2a \sum_{j \neq} q_j}{\left(q_i + \sum_{j \neq} q_j\right)^3} \leq 0 \tag{2}$$

which is always met. Imposing the symmetry condition $q_i = q$ for all $q_i = 1, 2, 3, ...N$, one obtains the Cournot-Nash equilibrium $q^{CN} = a\,(n-1)\,/\,(N^2 c)$, yielding profits $\pi^{CN} = a/N^2$. If the N firms were operating under perfect competition, then $p^* = c$ and therefore $q^* = a/\,(Nc)$.

It is apparent that the above solutions (i.e., both the Cournot-Nash equilibrium and the perfectly competitive equilibrium) are determinate for all $c > 0$, while they become indeterminate in correspondence of $c = 0$.

3 The dynamic setup

Here, I shall consider two well known capital accumulation rules. In both models, the market exists over $t \in [0\,, \infty)$, and is served by N firms producing a homogeneous good. Let $q_i(t)$ define the quantity sold by firm i at time t. The marginal production cost is constant and equal to c for all firms. Firms compete *à la* Cournot, the demand function at time t being:

$$p(t) = \frac{a}{Q(t)}\,,\; Q(t) = \sum_{i=1}^{N} q_i(t)\,;\; a > c. \tag{3}$$

In order to produce, firms must accumulate capacity or physical capital $k_i(t)$ over time. The two models I consider in the present paper are characterised by two different kinematic equations for capital accumulation.

A] The Solow (1956) or Swan (1956) setting, with the relevant dynamic equation being:

$$\frac{dk_i(t)}{dt} \equiv \dot{k}_i = I_i(t) - \delta k_i(t)\,, \tag{4}$$

where $I_i(t)$ is the investment carried out by firm i at time t, and δ is the constant depreciation rate. The instantaneous cost of investment is $C_i[I_i(t)] = bI_i^2(t)$, with $b > 0$. I also assume that firms operate with a constant returns technology and sell at full capacity at any time t, so that $q_i(t) = k_i(t)$. The demand function rewrites as:

$$p(t) = \frac{a}{\sum_{i=1}^{N} k_i(t)}\,. \tag{5}$$

Here, the control variable is the instantaneous investment $I_i(t)$, while the state variable is obviously $k_i(t)$.

B] The Ramsey (1928) setting, with the following dynamic equation:

$$\frac{dk_i(t)}{dt} \equiv \dot{k}_i = f(k_i(t)) - q_i(t) - \delta k_i(t)\,, \tag{6}$$

where $f(k_i(t)) = y_i(t)$ denotes the output produced by firm i at time t. In this setup, firms are assumed to use a decreasing return technology $f(k_i(t))$, with $f'(k_i(t)) \equiv \partial f(k_i(t))/\partial k_i(t) > 0$ and $f''(k_i(t)) \equiv$

$\partial^2 f(k_i(t))/\partial k_i(t)^2 < 0$. In this case, capital accumulates as a result of intertemporal relocation of unsold output $y_i(t) - q_i(t)$.[3] This can be interpreted in two ways. The first consists in viewing this setup as a corn-corn model, where unsold output is reintroduced in the production process. The second consists in thinking of a two-sector economy where there exists an industry producing the capital input which can be traded against the final good at a price equal to one (for further discussion, see Cellini and Lambertini, 2007). In this model, the control variable is $q_i(t)$, while the state variable remains $k_i(t)$. The demand function is (3).

In model [A], the problem of firm i is to choose the instantaneous investment $I_i(t)$ so as to maximisize its own discounted profits:

$$\Pi_i(\mathbf{k}(t), \mathbf{I}(t)) \triangleq \int_0^{\infty} \left\{[p(t) - c]\, k_i(t) - bI_i^2(t)\right\} e^{-\rho t} dt \tag{7}$$

s.t. the price dynamics (4) and the initial conditions $k_i(0) = k_{i0}$. $\mathbf{k}(t)$ and $\mathbf{I}(t)$ are the vector of all firms' states and controls, respectively.

In model [B], the problem of firm i is to choose the output level $q_i(t)$ so as to maximisize its own discounted profits:

$$\Pi_i(\mathbf{k}(t), \mathbf{q}(t)) \triangleq \int_0^{\infty} [p(t) - c]\, q_i(t)\, e^{-\rho t} dt \tag{8}$$

s.t. the price dynamics (6) and the initial conditions $k_i(0) = k_{i0}$. $\mathbf{k}(t)$ and $\mathbf{q}(t)$ are the vector of all firms' states and controls, respectively.

4 The Solow-Swan game

Given that the optimal control problem of firm i has not a linear-quadratic form, I will confine my attention to the open-loop solution. The Hamiltonian

[3] Of course, capacity decumulates whenever $y_i(t) - q_i(t) \leq 0$.

of firm i is:

$$\mathcal{H}_i(\mathbf{k}(t), \mathbf{I}(t)) = e^{-\rho t}\left\{\left[\frac{a}{k_i(t)+\sum_{j\neq i}k_j(t)}-c\right]k_i(t)-bI_i^2(t)\right. \tag{9}$$
$$\left.+\lambda_{ii}(t)[I_i(t)-\delta k_i(t)]+\sum_{j\neq i}\lambda_{ij}(t)[I_j(t)-\delta k_j(t)]\right\}$$

where $\lambda_{ij}(t) = \mu_{ij}(t)e^{\rho t}$, and $\mu_{ij}(t)$ is the co-state variable that firm i associates to $k_{ij}(t)$.

The necessary conditions are:[4]

$$\frac{\partial \mathcal{H}_i(\cdot)}{\partial I_i(t)} = -2bI_i(t) + \lambda_{ii}(t) = 0; \tag{10}$$

$$-\frac{\partial \mathcal{H}_i(\cdot)}{\partial k_i(t)} = \dot{\lambda}_{ii}(t) - \rho\lambda_{ii}(t) \Leftrightarrow$$

$$\dot{\lambda}_{ii}(t) = \lambda_{ii}(t)(\rho+\delta) + c - \frac{a\sum_{j\neq i}k_j(t)}{\left[k_i(t)+\sum_{j\neq i}k_j(t)\right]^2} \tag{11}$$

with the transversality condition

$$\lim_{t\to\infty} e^{-\rho t}\lambda_{ii}(t)k_i(t) = 0. \tag{12}$$

Note that the $N-1$ co-state equations pertaining to any $\lambda_{ij}(t)$, with $j \neq i$, are omitted as they are irrelevant due to the fact that the game exhibits separate state equations, i.e., the state dynamics of any firm is independent of the rivals' states and controls.

Before proceeding any further, it is worth noting that the first order condition (10) is altogether independent of the shape of the market demand function, since it is univocally determined by the marginal cost of instantaneous investment and the co-state variable pertaining to the capital accumulation

[4]Exponential discounting is omitted for brevity.

dynamics. This amounts to saying that, since the sales level is not a control (as it is instead in the static model outlined in section 2), here the convexity of the demand function may not cause the solution to be indeterminate.

Now, solving (10) w.r.t. $\lambda_{ii}(t)$, we obtain:[5]

$$\lambda_{ii} = 2bI_i \tag{13}$$

which can be differentiated w.r.t. time to yield the control equation:

$$\dot{I}_i = \frac{\dot{\lambda}_{ii}}{2b}. \tag{14}$$

Using (11) and (13), the above dynamics writes as follows:

$$\dot{I}_i = \frac{\left(k_i + \sum_{j\neq i} k_j\right)^2 \left[2bI_i(\rho+\delta)+c\right] - a\sum_{j\neq i} k_j}{2b\left(k_i + \sum_{j\neq i} k_j\right)^2} \tag{15}$$

that can be further simplified by introducing a symmetry assumption whereby $I_i = I$ and $k_i = k$ for all i :

$$\dot{I} = \frac{kN^2\left[2bI(\rho+\delta)+c\right] - a(N-1)}{2bkN^2}. \tag{16}$$

Imposing stationarity, $\dot{I} = 0$ yields the expression of the optimal steady state investment as a function of k :

$$I^{ss} = \frac{a(N-1) - ckN^2}{2bkN^2(\rho+\delta)} \tag{17}$$

which can be plugged into $\dot{k} = 0$ to obtain the steady state capacity endowment:[6]

$$k^{ss} = \frac{-cN + \sqrt{c^2N^2 + 8ab(N-1)(\rho+\delta)\delta}}{4bN(\rho+\delta)\delta}. \tag{18}$$

[5]Henceforth I will omit the explicit indication of the time argument.

[6]The second root can be disregarded as it is negative.

Using (18), the steady state investment rewrites:

$$I^{ss}=\frac{-cN+\sqrt{c^2N^2+8ab\left(N-1\right)\left(\rho+\delta\right)\delta}}{4bN\left(\rho+\delta\right)}. \tag{19}$$

The corresponding equilibrium price and profits are:

$$p^{ss}=\frac{a}{Nk^{ss}}=\frac{4ab\left(\rho+\delta\right)\delta}{-cN+\sqrt{c^2N^2+8ab\left(N-1\right)\left(\rho+\delta\right)\delta}}; \tag{20}$$

$$\begin{aligned}\pi^{ss} &= \left\{Nc\left(2\rho+\delta\right)\left(cN-\sqrt{c^2N^2+8ab\left(N-1\right)\left(\rho+\delta\right)\delta}\right)+\right. \\ &\left. 4ab\left(\rho+\delta\right)\delta\left[2N\rho+\delta\left(N+1\right)\right]\right\}/\left[8bN^2\left(\rho+\delta\right)^2\delta\right].\end{aligned} \tag{21}$$

It is immediate to check that the following properties hold, in the limit, as marginal cost c tends to zero:

$$\lim_{c\to 0}k^{ss}=\frac{a\left(N-1\right)}{N\sqrt{2ab\left(N-1\right)\left(\rho+\delta\right)\delta}};\ \lim_{c\to 0}I^{ss}=\delta\lim_{c\to 0}k^{ss} \tag{22}$$

$$\lim_{c\to 0}p^{ss}=\frac{ab\left(\rho+\delta\right)\delta\sqrt{2}}{\sqrt{ab\left(N-1\right)\left(\rho+\delta\right)\delta}};\ \lim_{c\to 0}\pi^{ss}=\frac{a\left[2N\rho+\delta\left(N+1\right)\right]}{2N^2\left(\rho+\delta\right)} \tag{23}$$

On the basis of (22-23), without further proof, I can state:

Proposition 1 *The equilibrium of the Solow-Swan game is determinate for all* $N>1$ *and all* $c\geq 0$.

Now I may proceed to evaluate the stability properties of the steady state equilibrium identified by the pair $\left(k^{ss},I^{ss}\right)$. This is done in the following:

Proposition 2 *The steady state* $\left(k^{ss},I^{ss}\right)$ *is a saddle point for all* $c\geq 0$.

Proof. The stability properties of the dynamic system (4-16) can be assessed by evaluating the trace and determinant of the associated Jacobian matrix:

$$\Psi=\begin{bmatrix}\frac{\partial\dot{k}}{\partial k} & \frac{\partial\dot{k}}{\partial I}\\ \frac{\partial\dot{I}}{\partial k} & \frac{\partial\dot{I}}{\partial I}\end{bmatrix} \tag{24}$$

At (k^{ss}, I^{ss}), the trace and determinant of matrix Ψ are:

$$\begin{aligned} T(\Psi) &= \rho \\ \Delta(\Psi) &= -\frac{a(N-1)}{2bN^2k^2} - \delta(\rho+\delta) < 0 \,\forall\, c \geq 0. \end{aligned} \tag{25}$$

This proves the claim. ■

5 The Ramsey game

As in the previous case, also here, given that the optimal control problem of firm i has not a linear-quadratic form, I will confine my attention to the open-loop solution. The Hamiltonian of firm i is:

$$\begin{aligned} \mathcal{H}_i(\mathbf{k}(t), \mathbf{q}(t)) \;=\; & e^{-\rho t}\left\{\left[\frac{a}{q_i(t)+\sum_{j\neq i} q_j(t)} - c\right] q_i(t)\right. \\ & + \lambda_{ii}(t)\left[f(k_i(t)) - q_i(t) - \delta k_i(t)\right] \\ & \left. + \sum_{j\neq i} \lambda_{ij}(t)\left[f(k_j(t)) - q_j(t) - \delta k_j(t)\right]\right\} \end{aligned} \tag{26}$$

where $\lambda_{ij}(t) = \mu_{ij}(t)\, e^{\rho t}$, $\mu_{ij}(t)$ being the co-state variable that firm i associates to $k_{ij}(t)$.

The first order condition on control $q_i(t)$ is:[7]

$$\frac{\partial \mathcal{H}_i(\cdot)}{\partial q_i(t)} = \frac{a\sum_{j\neq} q_j(t)}{\left[q_i(t)+\sum_{j\neq} q_j(t)\right]^2} - c - \lambda_{ii}(t) = 0; \tag{27}$$

$$-\frac{\partial \mathcal{H}_i(\cdot)}{\partial k_i(t)} = \dot{\lambda}_{ii}(t) - \rho\lambda_{ii}(t) \Leftrightarrow$$

$$\dot{\lambda}_{ii}(t) = -\lambda_{ii}(t)\left[f'(k_i(t)) - \rho - \delta\right] \tag{28}$$

[7] Again, exponential discounting is omitted for brevity.

with the transversality condition

$$\lim_{t\to\infty} e^{-\rho t}\lambda_{ii}(t)\, k_i(t) = 0. \tag{29}$$

Similarly to the Solow-Swan game, the $N-1$ co-state equations pertaining to any $\lambda_{ij}(t)$, with $j \neq i$, are omitted as they are irrelevant due to the fact that the game exhibits separate state equations, i.e., the state dynamics of any firm is independent of the rivals' states and controls.

Comparing (27) and (1), one immediately sees that the presence of capital accumulation in the dynamic game plays a key role in opening the way towards a solution to the indeterminacy issue affecting the static game as marginal cost c tends to zero, precisely because of the fact that the co-state variable that firm i attaches to its own capacity accumulation dynamics enters the FOC on the investment control.[8]

From (27), one obtains the expression of the co-state variable $\lambda_{ii}(t)$:[9]

$$\lambda_{ii} = \frac{a\sum_{j\neq} q_j(t)}{\left[q_i(t) + \sum_{j\neq} q_j(t)\right]^2} - c. \tag{30}$$

Then, differentiating the above expression w.r.t. time yields:

$$\dot{\lambda}_{ii} = \frac{a\sum_{j\neq} \dot{q}_j}{\left[q_i(t) + \sum_{j\neq} q_j(t)\right]^2} - \frac{2a\sum_{j\neq} q_j\left(\dot{q}_i + \sum_{j\neq} \dot{q}_j\right)}{\left[q_i(t) + \sum_{j\neq} q_j(t)\right]^3} \tag{31}$$

which, using (30) and imposing symmetry across control, state and co-state

[8] To this regard, it is worth stressing that the co-state variable appearing in the open-loop formulation of the game cannot be properly considered as a shadow price (of an additional unit of capacity, in the present setup), as it would instead be true for the first partial derivative of the value function appearing in the Bellman equation of the corresponding feedback problem (see Caputo, 2007).

[9] In the remainder of the section, I will omit the explicit indication of the time argument.

variables, yields the following control dynamics:

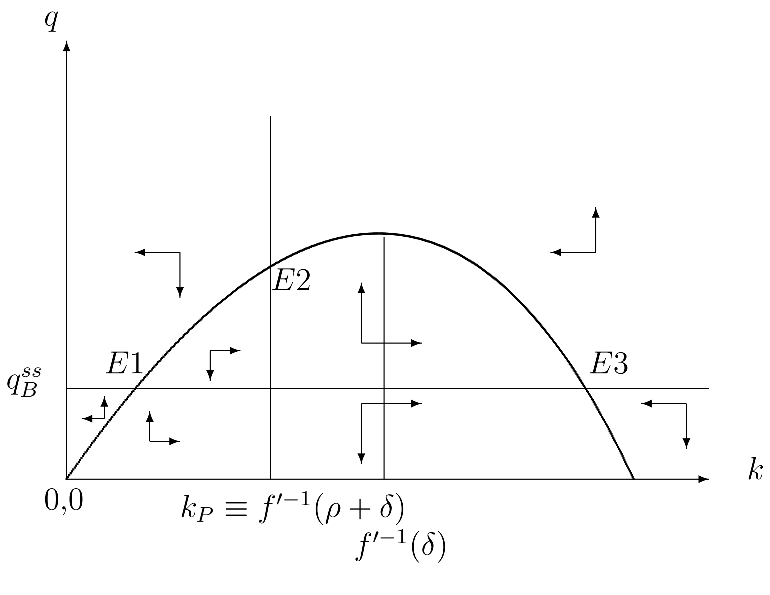
$$\dot{q} = \frac{q\left[a\left(N-1\right)-cN^2q\right]\left[f'(k)-\rho-\delta\right]}{a\left(N-1\right)}. \tag{32}$$

The stationarity condition $\dot{q}=0$ is satisfied by

$$q_A^{ss}=0\,;\; q_B^{ss}=\frac{a\left(N-1\right)}{cN^2}\,;\; f'(k)=\rho+\delta, \tag{33}$$

where (i) q_A^{ss} implies that firms don't sell, and therefore their equilibrium profits are obviously nil; q_B^{ss} coincides with the solution of the static game illustrated in section 2, and therefore is an acceptable solution only if marginal cost c is strictly positive; and $f'(k)=\rho+\delta$ is the *Ramsey golden rule*. As we are about to see, only the second and third roots of $\dot{q}=0$ are relevant, while the first one can be disregarded.

Figure 1: The phase diagram



The phase diagram of the present model can be drawn in the space $\{k, q\}$, as in Figure 1. The locus $\dot{q} \equiv dq/dt = 0$ is given by the solutions in (33). Putting aside the horizontal axis corresponding to $q_A^{ss} = 0$, the two remaining loci partition the space $\{k, q\}$ into four regions, where the dynamics of q is summarised by the vertical arrows. The locus $\dot{k} \equiv dk/dt = 0$ as well as the dynamics of k, depicted by the horizontal arrows, derive from (6). Steady state equilibria, denoted by $E1$, $E2$ along the horizontal branch, and $E3$ along the vertical one, are identified by the intersections between loci.

Figure 1 describes only one out of five possible configurations, due to the fact that the position of the vertical line $f'(k) = \rho + \delta$ is independent of demand parameters, while the locus $q_B^{ss} = a\,(N-1)\,/\,(cN^2)$ shifts upwards (resp., downwards) as a (resp., c) increases. Therefore, we obtain one out of five possible regimes:

1. There exist three steady state points, with $k_{E1} < k_{E2} < k_{E3}$ (this is the specific case portrayed in Figure 1).

2. There exist two steady state points, with $k_{E1} = k_{E2} < k_{E3}$.

3. There exist three steady state points, with $k_{E2} < k_{E1} < k_{E3}$.

4. There exist two steady state points, with $k_{E2} < k_{E1} = k_{E3}$.

5. There exists a unique steady state equilibrium point, corresponding to $E2$.

The vertical locus $f'(k) = \rho + \delta$ is a constraint on optimal capital, determined by firms' intertemporal preferences, i.e., their common discount rate, and depreciation. This is the *Ramsey optimal capital endowment* k^R. When market size a (resp., marginal cost c) is very large (resp., low), points $E1$

and $E3$ either do not exist (regime 5) or fall to the right of $E2$ (regimes 2, 3 and 4). In such a case, the capital constraint is operative and firms choose the capital accumulation corresponding to $E2$.

Notice that, as $E1$ and $E3$ entail the same levels of sales, point $E3$ is surely inefficient in that it requires a higher amount of capital. $E1$ corresponds to the optimal quantity emerging from the static version of the game.

The stability analysis of the above system reveals that:

Regime 1. $E1$ is a saddle point, while $E2$ is an unstable focus. $E3$ is again a saddle point, with the horizontal line as the stable manifold.

Regime 2. $E1$ coincides with $E2$, so that we have only two steady states which are both are saddle points. In $E1 = E2$, the saddle path approaches the saddle point from the left only, while in $E3$ the stable manifold is again the horizontal line.

Regime 3. $E2$ is a saddle, $E1$ is an unstable focus. $E3$ is a saddle point, as in regimes 1 and 2.

Regime 4. Here, $E1$ and $E3$ coincide. $E3$ remains a saddle, while $E1 = E3$ is a saddle whose converging manifold proceeds from the right along the horizontal line.

Regime 5. Here, there exists a unique steady state point, $E2$, which is a saddle point.

Residually, the dynamics illustrated in Figure 1 intuitively reveals that the origin (point $(0, 0)$) is unstable.

We can sum up the above discussion as follows. The unique efficient and non-unstable steady state point is $E2$ if $k_{E2} < k_{E1}$, while it is $E1$ if the opposite inequality holds. Such a point is always a saddle. Individual equilibrium output is q_B^{ss} if the equilibrium is in $E1$, or $q^R\left(k^R\right) = f\left(k^R\right) - \delta k^R$ (i.e., the output level corresponding to the optimal capital constraint k^R) if the equilibrium is point $E2$. The reason is that, if the capacity at which marginal instantaneous profit is nil is larger than the optimal capital constraint, the latter becomes binding. Otherwise, the capital constraint is irrelevant, and firms' decisions in each period are driven by the unconstrained maximisation of single-period profits only.

The foregoing discussion allows me to state, without further proof, the following result:

Proposition 3 *The Ramsey game yields a saddle point equilibrium where individual output is*

$$q^{ss} = \min\left\{\frac{a\,(N-1)}{cN^2}\,;\, f\left(k^R\right) - \delta k^R\right\}$$

for all $c > 0$.

The above Proposition has a relevant Corollary:

Corollary 4 *In the limit, as* c *tends to zero, the Ramsey game reaches a saddle point equilibrium in correspondence of the golden rule where* $q^R\left(k^R\right) = f\left(k^R\right) - \delta k^R$.

The proof of this ancillary result is intuitive, as it follows immediately from the observation that $\lim_{c\to 0} q_B^{ss} = \infty$; accordingly, in such a case the horizontal branch does not intersect the locus $\dot{k} = 0$ any more and the only stable solution remaining is that where $f'(k) = \rho + \delta$.

6 Feedback solutions

A natural extension consists in considering the feedback solution of the two games outlined above, attained through the Bellman equation approach. As has been already stated, the Bellman equations arising in both cases cannot be solved analytically since the games at hand are not defined in a linear-quadratic form. However, a relevant implication of the first order conditions can be easily drawn.

Let's start with the Bellman equation of firm i in the Solow-Swan setup:

$$\rho V_i(\mathbf{k}(t)) = \max_{I_i(t)} \left\{ \left[\frac{a}{k_i(t) + \sum_{j \neq i} k_j(t)} - c \right] k_i(t) - b I_i^2(t) + \right.$$

$$\left. V_{ii}'(\mathbf{k}(t)) [I_i(t) - \delta k_i(t)] + \sum_{j \neq i} V_{ij}'(\mathbf{k}(t)) [I_j(t) - \delta k_j(t)] \right\} \qquad (34)$$

where $V_i(\mathbf{k}(t))$ is the value function and $V_{ij}'(\mathbf{k}(t)) = \partial V_i(\mathbf{k}(t)) / \partial k_j(t)$ for all i and j. Now, taking the first order condition, we have:

$$-2bI_i(t) + V_{ii}'(\mathbf{k}(t)) = 0, \qquad (35)$$

whose solution, as in the open-loop case (see eq. (10)), is independent of the marginal cost c, except that $V_{ii}'(\mathbf{k}(t))$ replaces $\lambda_{ii}(t)$.

Turning to the Ramsey model, we have the following Bellman equation for firm i:

$$\rho V_i(\mathbf{k}(t)) = \max_{q_i(t)} \left\{ \left[\frac{a}{q_i(t) + \sum_{j \neq i} q_j(t)} - c \right] q_i(t) + \right. \qquad (36)$$

$$\left. V_{ii}'(\mathbf{k}(t)) [f(k_i(t)) - q_i(t) - \delta k_i(t)] + \sum_{j \neq i} V_{ij}'(\mathbf{k}(t)) [f(k_j(t)) - q_j(t) - \delta k_j(t)] \right\}.$$

The FOC taken w.r.t. $q_i(t)$ is:

$$\frac{a\sum_{j\neq} q_j(t)}{\left[q_i(t) + \sum_{j\neq} q_j(t)\right]^2} - c - V'_{ii}\left(\mathbf{k}\left(t\right)\right) = 0, \tag{37}$$

and once again the solution exists at any instant t for all $c \in [0, a)$, provided $V'_{ii}\left(\mathbf{k}\left(t\right)\right)$ is not nil (which can be ruled out, in general, as it would entail that the shadow price of an additional unit of productive capacity is nil).

7 Concluding remarks

I have revisited the Cournot oligopoly with isoelastic demand function using a dynamic approach based upon two different capital accumulation dynamics, based on Solow (1956) - Swan (1956) and Ramsey, respectively. In both cases, the presence of capacity accumulation eliminates a well known drawback affecting the static approach to the solution of Cournot games with hyperbolic demand, namely, the indeterminacy of equilibrium when marginal production costs are nil. The solution of this long standing issue is a direct consequence of two facts i.e., that (i) in the Solow-Swan approach quantity is no longer a control but a state variable; and (ii) in the Ramsey approach, although quantity is indeed a control, there exists a steady state solution which is independent of marginal costs (as well as demand parameters) and is determined by the interplay between intertemporal parameters and the marginal productivity of capital, i.e., the Ramsey golden rule.

References

[1] Anderson, S.P. and M.P. Engers (1992), “Stackelberg versus Cournot Oligopoly Equilibrium”, *International Journal of Industrial Organization*, **10**, 127-35.

[2] Anderson, S.P. and M.P. Engers (1994), “Strategic Investment and Timing of Entry”, *International Economic Review*, **35**, 833-53.

[3] Calzolari, G. and L. Lambertini (2006), “Tariffs vs Quotas in a Trade Model with Capital Accumulation”, *Review of International Economics*, **14**, 632-44.

[4] Calzolari, G. and L. Lambertini (2007), “Export Restraints in a Model of Trade with Capital Accumulation”, *Journal of Economic Dynamics and Control*, **31**, 3822-42.

[5] Caputo, M.R. (2007), “The Envelope Theorem for Locally Differentiable Nash Equilibria of Finite Horizon Differential Games”, *Games and Economic Behavior*, **61**, 198-224.

[6] Cellini, R. and L. Lambertini (1998), “A Dynamic Model of Differentiated Oligopoly with Capital Accumulation”, *Journal of Economic Theory*, **83**, 145-55.

[7] Cellini, R. and L. Lambertini (2007), “Capital Accumulation, Mergers, and the Ramsey Golden Rule”, in M. Quincampoix, T. Vincent and S. Jørgensen (eds), *Advances in Dynamic Game Theory and Applications*, Annals of the International Society of Dynamic Games, vol. 8, Boston, Birkhäuser, 487-505.

[8] Dixit, A.K. (1986), "Comparative Statics for Oligopoly", *International Economic Review*, **27**, 107-22.

[9] Friedman, J.W. (1977), *Oligopoly and the Theory of Games*, Amsterdam, North-Holland.

[10] Ramsey, F.P. (1928), "A Mathematical Theory of Saving", *Economic Journal*, **38**, 543-59. Reprinted in Stiglitz, J.E. and H. Uzawa (1969, eds.), *Readings in the Modern Theory of Economic Growth*, Cambridge, MA, MIT Press.

[11] Solow, R. (1956), "A Contribution to the Theory of Economic Growth", *Quarterly Journal of Economics*, **70**, 65-94.

[12] Swan, T.W. (1956), "Economic Growth and Capital Accumulation", *Economic Record*, **32**, 334-61.